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LETTER TO THE EDITOR

Yang–Baxterization of reflection equations for $\check{R}$ with two distinct eigenvalues

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Abstract. A condition of Yang–Baxterization of reflection equations for $\check{R}$ with two distinct eigenvalues is given. Constant solutions to reflection equations for $\check{R}$ related to fundamental representations of $sl_q(n)$ and $sl_q(1,1)$ presented in [6] and for $\check{R}$ of the eight-vertex model are Yang–Baxterized.

Reflection equations (RE) were introduced in [1] as an equation describing factoring scattering on a halfline. Recently they also appeared in quantum current algebras [2] and in integrable models with non-periodic boundary conditions [3, 4]. Kulish *et al* proposed the concept of reflection algebras (RA) related to RE [5], studied their properties (quantum-group-comodule property) and constructed constant solutions of RE (i.e. one-dimensional representations of RA) [6]. RA related to the eight-vertex model is formulated in [7]. Papers [8–10] were devoted to representations of RA.

As far as we know, however, the problem of Yang–Baxterization of RE has not been touched so far. In this letter we shall investigate Yang–Baxterization of the following RE without spectral parameters

$$RK_1\tilde{R}K_2 = K_2RK_1\tilde{R} \tag{1}$$

where $K_1 = K\otimes I$, $K_2 = I\otimes K$, $\tilde{R} = PRP$, and P is the permutation operator $P(x\otimes y) = y\otimes x$, for $\check{R} = PR$ having two distinct eigenvalues, namely, incorporate λ-dependence into $\check{R}$ and K such that they satisfy the following spectral-parameter-dependent RE

$$R(\lambda\mu^{-1})K_1(\lambda)\tilde{R}(\lambda\mu)K_2(\mu) = K_2(\mu)R(\lambda\mu)K_1(\lambda)\tilde{R}(\lambda\mu^{-1}). \tag{2}$$

In terms of $\check{R} = PR$, equations (1) and (2) are rewritten as

$$\check{R}K_1\check{R}K_1 = K_1\check{R}K_1\check{R} \tag{3}$$

$$\check{R}(\lambda\mu^{-1})K_1(\lambda)\check{R}(\lambda\mu)K_1(\mu) = K_1(\mu)\check{R}(\lambda\mu)K_1(\lambda)\check{R}(\lambda\mu^{-1}) \tag{4}$$

where $PK_2P = K_1$ is used.

Yang–Baxterization of Yang–Baxter equations has been extensively studied [11, 12]. For $\check{R}$ with two distinct eigenvalues t_1 and t_2, the explicit Yang–Baxterization for $\check{R}$ is given by

$$\check{R}(\lambda)=t_2^{-1}\lambda^{-1}\check{R}+t_1\lambda\check{R}^{-1}. \tag{5}$$

By using $\check{R}+t_1t_2\check{R}^{-1}=t_1+t_2$, equation (5) can be rewritten as (up to a constant)

$$\check{R}(\lambda)=(\lambda-\lambda^{-1})\check{R}-\lambda(t_1+t_2)I. \tag{6}$$

These $R(\lambda)=P\check{R}(\lambda)$ satisfy the Yang–Baxter equation with parameters. Therefore, what we need to do is to Yang–Baxterize the K-matrix.

We start with an instructive example. Consider the $\check{R}$-matrix

$$\check{R}=\begin{pmatrix} q & & & \\ & 0 & 1 & \\ & 1 & \omega & \\ & & & q \end{pmatrix} \qquad \omega=q-q^{-1}. \tag{7}$$

Letting

$$K=\begin{pmatrix} z & y \\ x & u \end{pmatrix} \tag{8}$$

and substituting equations (7) and (8) into (3) we have

$$\begin{aligned} &ux=q^{-2}xu \qquad [u,z]=0, \qquad [x,z]=-q^{-1}\omega ux \\ &uy=q^2yu \qquad [x,y]=q^{-1}\omega(uz-u^2) \qquad [y-z]=q^{-1}\omega yu. \end{aligned} \tag{9}$$

The associative algebra over $\mathbb{C}$ ($\mathbb{C}$ is the complex number field) generated by x, y, z, u and the unit 1 subject to relations (6) is the RA $\mathcal{A}_2$ (in Kulish's notation). This algebra has two central elements

$$C_1=u+q^2z \qquad C_2=\det_qK=uz-q^2yx.$$

It is easy to prove that K has an inverse

$$K^{-1}=C_2^{-1}(C_1I-q^2K). \tag{10}$$

Now we prove a key relation in Yang–Baxterization

$$C_2^2[R,K_1^{-2}]+q^4[R,K_1^2]=0. \tag{11}$$

In fact, from equation (10) it follows that

$$-q^2K_1^2+C_1K_1=C_2 \tag{12}$$

$$K_1^{-2}=C_2^{-2}(C_1^2I-2C_1q^2K_1+q^4K_1^2) \tag{13}$$

which leads to relation (11) immediately.

For the case in hand, $t_1=-q^{-1}$, $t_2=q$, the Yang–Baxterization for $\check{R}$ is

$$\check{R}(\lambda)=\lambda^{-1}\check{R}-\lambda\check{R}^{-1}. \tag{14}$$

Suppose that

$$K(\lambda)=K+\alpha(\lambda)C_2K^{-1} \tag{15}$$

where $\alpha(\lambda)$ is to be determined such that $\check{R}(\lambda)$ and $K(\lambda)$ satisfy the RE (4). Substituting (14) and (15) into (4), we find that

$$\alpha(\lambda)=\pm q^{-2}\lambda^2 \tag{16}$$

where equation (3) and the relation $\check{R}-\check{R}^{-1}=\omega I$ are used. So the Yang–Baxterization for K is obtained as

$$K(\lambda)=K+\alpha\lambda^2 C_2 K^{-1} \qquad \alpha=\pm q^{-2} \tag{17}$$

or, equivalently, as

$$K(\lambda)=\lambda^{-1}K+\alpha\lambda C_2 K^{-1}. \tag{18}$$

We see that the quantum determinant of RA plays an important role in above example. However, for any R-matrix the quantum determinant of RA does not keep the central elements. For example, the RA related to the *non-standard* $\check{R}$

$$\check{R}=\begin{pmatrix} q & & & \\ & 0 & 1 & \\ & 1 & \omega & \\ & & & -q^{-1} \end{pmatrix} \qquad \omega=q-q^{-1} \tag{19}$$

has only one central element $u-z$ in the case $q^2\neq-1$ (see next section) and the determinant of its constant solutions is vanishing. How does one deal with this situation? The key point is to avoid using K^{-1}. In above example, the $\check{R}(x)$ and $K(\lambda)$ can be rewritten as (choosing $\alpha=-q^{-2}$)

$$\begin{aligned} \check{R}(\lambda)&=(\lambda-\lambda^{-1})\check{R}-\omega\lambda I \\ K(\lambda)&=(\lambda-\lambda^{-1})K-q^{-2}\lambda C_1 I. \end{aligned} \tag{20}$$

This suggests that, for any $\check{R}$ with two distinct eigenvalues, which is Yang–Baxterized as (6), we could suppose that

$$K(\lambda)=(\lambda-\lambda^{-1})K+\lambda A I \tag{21}$$

where A is a central element of RA in the algebraic form of K or a constant in the constant solutions. The following proposition gives a condition of Yang–Baxterization:

Proposition 1. The reflection equation can be Yang–Baxterized through (21) if and only if

$$[\check{R}, K_1^2+AK_1]=0. \tag{22}$$

Proof. Substituting (6) and (21) into (4) we find that

$$(t_1+t_2)(\lambda\mu^{-1}-\lambda^{-1}\mu)(\lambda-\lambda^{-1})(\mu-\mu^{-1})\lambda\mu[\check{R}, K_1^2+AK_1]=0$$

from which we obtain (22). Here we have used the relation $t_1+t_2\neq 0$ (the case $t_1+t_2=0$ is trivial).

The proposition implies that if one can choose a central element A (or a constant) satisfying (21) then RE can be Yang–Baxterized as (21). We would like to point out that the condition (22) is not so rigid that all the constant solutions related to fundamental representations of $sl_q(n)$ and $sl_q(1,1)$ presented in [6] satisfy this condition, or, accurately speaking, satisfy

$$K_1^2+AK_1=BI \qquad (A, B\in\mathbb{C}).$$

For the above example, from (12) we find that $A=-q^{-2}C_1$, then we rederive (20).

We now turn to the example (19). If we suppose K takes the form (8), then we obtain

$$ux = q^2 xu \qquad [u, z] = 0 \qquad [x, z] = -q^{-1}\omega ux$$

$$uy = q^{-2} yu \qquad qxy + q^{-1}yx = \omega(uz - u^2) \qquad [y, z] = q^{-1}\omega yu \quad (23)$$

$$(q + q^{-1})x^2 = (q + q^{-1})y^2 = 0. \quad (24)$$

Relation (24) has a different form in different cases.

Case I. $q^2 = -1$. In this case (24) is automatically satisfied and the algebraic relations (23) are the same as (9) for $q^{-2} = -1$. Therefore it can be Yang–Baxterized through (18) or through (20).

Case II. If $q^2 \neq -1$, equation (24) becomes

$$x^2 = y^2 = 0. \quad (25)$$

In this case the RA has only one central element $u - z$. The constant solutions are

$$K^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad K^{(2)} = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix} \qquad (z \in \mathbb{C}).$$

The determinant of $K^{(2)}$ is vanishing, however, it satisfies

$$(K_1^{(2)})^2 - zK_1^{(2)} = 0$$

therefore it can be Yang–Baxterized as ($A = -z$)

$$K^{(2)}(\lambda) = (\lambda - \lambda^{-1})K^{(2)} - z\lambda I.$$

Paper [6] investigated a class of constant solutions to RE related to the fundamental representations of $sl_q(n)$. For $sl_q(3)$ there are two constant solutions with non-vanishing determinants

$$K^{(1)} = \begin{pmatrix} a_{11} & & \\ & a_{11} & \\ & & a_{11} \end{pmatrix} \qquad K^{(2)} = \begin{pmatrix} & & a_{13} \\ & a_{22} & \\ g_{31} & & a_{33} \end{pmatrix}$$

where $g_{31} = a_{22}(a_{22} - a_{33})/a_{13}$. It is easy to verify that

$$(K^{(1)})^2 + a_{11}K^{(1)} = 2a_{11}^2 I$$

$$(K^{(2)})^2 - a_{33}K^{(2)} = a_{13}g_{31}I$$

therefore they can be Yang–Baxterized as

$$K^{(1)}(\lambda) = (\lambda - \lambda^{-1})K^{(1)} + a_{11}\lambda I$$

$$K^{(2)}(\lambda) = (\lambda - \lambda^{-1})K^{(2)} - a_{33}\lambda I.$$

Here we would like to point out that the choice of A is not unique. For $K^{(1)}$ we also have

$$(K^{(1)})^2 - a_{11}K^{(1)} = 0$$

then the corresponding Yang–Baxterization is

$$K^{(1)}(\lambda) = (\lambda - \lambda^{-1})K^{(1)} - a_{11}\lambda I.$$

For $sl_q(4)$, besides the unity solution $K^{(1)}$, [6] presents the following two constant solutions

$$K^{(2)}=\begin{pmatrix} & & & a_{14} \\ & & a_{23} & \\ & a_{32} & a_{33} & \\ g_{41} & & & a_{33} \end{pmatrix} \qquad K^{(3)}=\begin{pmatrix} & & & a_{14} \\ & a_{22} & & \\ & & a_{22} & \\ a_{41} & & & g_{44} \end{pmatrix}$$

where $g_{41}=a_{23}a_{32}/a_{14}$, $g_{44}=a_{22}-(a_{14}a_{41})/a_{22}$. One can check that

$$(K^{(2)})^2-a_{33}K^{(2)}=a_{23}a_{32}I$$
$$(K^{(3)})^2-g_{44}K^{(3)}=a_{14}a_{41}I$$

then their Yang–Baxterizations are

$$K^{(2)}(\lambda)=(\lambda-\lambda^{-1})K^{(2)}-a_{33}\lambda I$$
$$K^{(3)}(\lambda)=(\lambda-\lambda^{-1})K^{(3)}-g_{44}\lambda I.$$

For $sl_q(n)$, except the unity matrix $K^{(1)}$, there exists another constant solution $K^{(2)}$ with matrix elements

$$K^{(2)}_{ij}=\delta_{i\,n+1-i}.$$

This solution satisfies $(K^{(2)})^2=I$, therefore its Yang–Baxterization is

$$K^{(2)}(\lambda)=(\lambda-\lambda^{-1})K^{(2)}.$$

We now consider another important example, the eight-vertex model. In this case the $\check{R}$-matrix reads (with eigenvalues $t_1=1-t$, $t_2=1+t$)

$$\check{R}=\begin{pmatrix} 1 & & & t \\ & 1 & \omega t & \\ & \omega t & 1 & \\ t & & & 1 \end{pmatrix}, \tag{26}$$

where $\omega^2=1$. The constant solutions are obtained as

$$K^{(1)}=\begin{pmatrix} x & y \\ \omega y & x \end{pmatrix} \qquad K^{(2)}=\begin{pmatrix} x & y \\ y & \omega x \end{pmatrix} \qquad x,y\in\mathbb{C}$$

which satisfy

$$(K^{(1)})^2-2xK^{(1)}=(\omega y^2-x^2)I$$
$$(K^{(2)})^2-(1+\omega)xK^{(2)}=(y^2-\omega x^2)I$$

therefore, their Yang–Baxterization are

$$K^{(1)}(\lambda)=(\lambda-\lambda^{-1})K^{(1)}-2x\lambda I$$
$$K^{(2)}(\lambda)=(\lambda-\lambda^{-1})K^{(2)}-(1+\omega)\lambda xI.$$

We have already established the theory of Yang–Baxterization of reflection equations for $\check{R}$ with two distinct eigenvalues. If one can prove that a solution to RE satisfies condition (22), then it can be Yang–Baxterized through relation (21). Although all the constant solutions in [6] satisfy this condition, we do not know if this condition is valid for any solutions to RE related to $\check{R}$ with two distinct eigenvalues. This is still an open problem. In further discussions, we shall consider the problem of Yang–Baxterization of RE related to $\check{R}$ with *three* and *four* distinct eigenvalues.

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